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Decomposition of Balanced Matrices.
Part IV: Connected Squares

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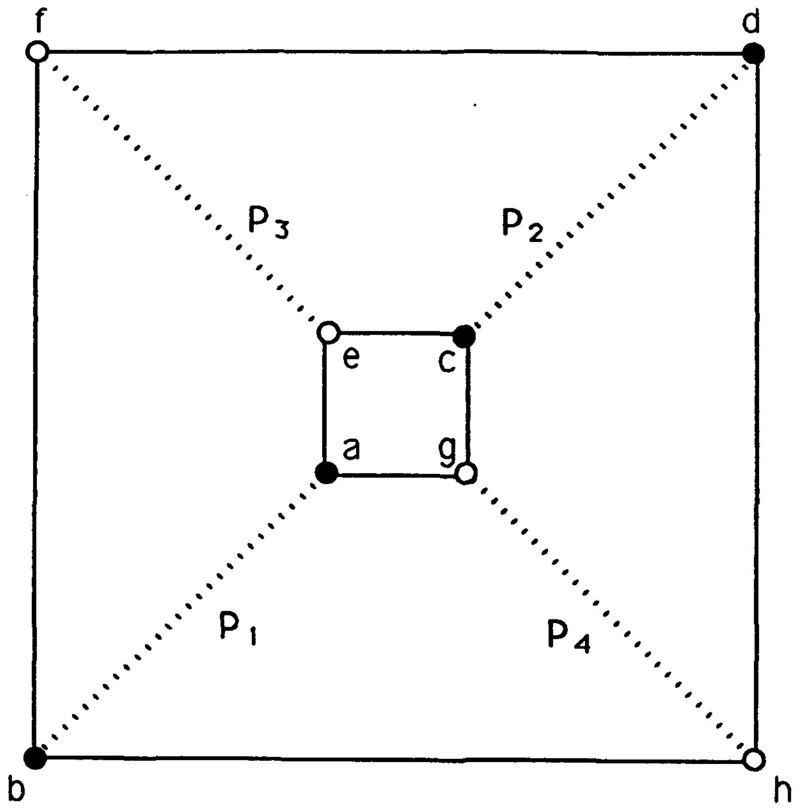

Figure 1: Connected squares

1 The Main Result

In this part we prove the following result:

Theorem 1.1 *Let G be a wheel-free bipartite graph which is signable to be balanced and contains connected squares. If the graph G has no biclique cutset, then there exist some connected squares $\Sigma = CS(P_1, P_2, P_3, P_4)$ and a 2-join, separating $V(P_1) \cup V(P_2)$, from $V(P_3) \cup V(P_4)$.*

We consider connected squares $CS(P_1, P_2, P_3, P_4)$ in a wheel-free bipartite graph G which is signable to be balanced and we define $\tilde{P}_i$, $1 \leq i \leq 4$ to be the subpath obtained from P_i by removing its endnodes. We assume that $a, b, c, d \in V^c$ and $e, f, g, h \in V^r$ and we use the notation of Figure 1.

2 A Classification of Nodes and Paths

The following theorem characterizes the strongly adjacent nodes to connected squares $\Sigma = CS(P_1, P_2, P_3, P_4)$.

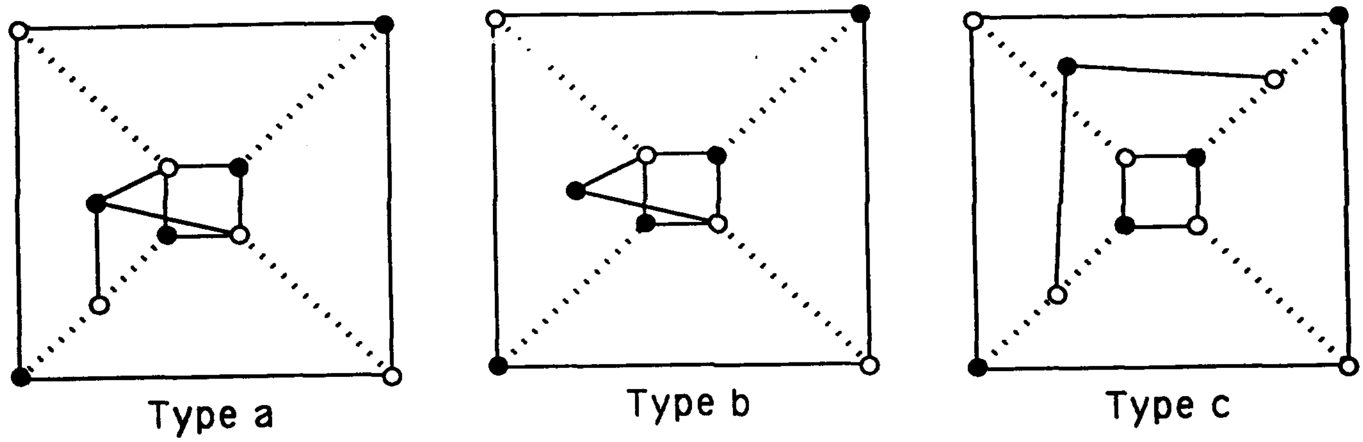

Figure 2: Strongly adjacent nodes

Theorem 2.1 *Let v be a strongly adjacent node to connected squares $\Sigma = CS(P_1, P_2, P_3, P_4)$. Then one of the following holds:*

- *Node v has exactly two neighbors in Σ, both contained in P_i, for $i = 1, 2, 3, 4$.*
- *Node v is of one of the following types, see Figure 2:*

  Type a *Node v has three neighbors in Σ, two of them being a, c or e, g or b, d or f, h. If $v \in V^c$, the third neighbor is in $\tilde{P}_1$ or in $\tilde{P}_2$. If $v \in V^r$, the third neighbor is in $\tilde{P}_3$ or in $\tilde{P}_4$.*

  Type b *Node v has exactly two neighbors in Σ which are a, c or e, g or b, d or f, h.*

  Type c *Node v has exactly two neighbors in Σ and if $v \in V^c$, then v has one neighbor in $\tilde{P}_1$ and one in $\tilde{P}_2$. If $v \in V^r$, then v has one neighbor in $\tilde{P}_3$ and one in $\tilde{P}_4$.*

Proof: Let w be a strongly adjacent node to Σ, and assume w.l.o.g. that $w \in V^c$. Then w cannot be adjacent to all the nodes in the set $\{e, f, g, h\}$ otherwise w is the center of a wheel. This implies that w cannot have neighbors in all the paths P_1, P_2, P_3, P_4, otherwise, assume w.l.o.g. that w is not adjacent to e, then there is a $3PC(w, e)$.

Now assume w.l.o.g. that w has no neighbors in P_4 and consider the parachute Π having a, g, c as short top, a, P_1, b, f and c, P_2, d, f as long sides and P_3 as middle path. Then $N(w) \cap V(\Sigma) = N(w) \cap V(\Pi)$. Hence we apply Theorem 2.1(III) to the parachute Π. The first case of the above theorem corresponds to the first case of Theorem 2.1(III). If node w is of

Type a[2.1(III)], then w is of Type b in this theorem and if node w is of Type b[2.1(III)], then w is of Type c in this theorem. Node w cannot be of Type c[2.1(III)], else f is the center of an odd wheel. Node w cannot be of Type d[2.1(III)], else there is a $3PC(h,a)$ or a $3PC(h,c)$.

Furthermore w cannot be of Type e[2.1(III)] or of Type h[2.1(III)] or of Type i[2.1(III)] or of Type o[2.1(III)], else w has a neighbor in P_4. If w is of Type g[2.1(III)], then w is a twin of a node in Σ.

We finally examine the case in which w is of Type f[2.1(III)]. Let w_1 be the neighbor of w in $V(P_1)\setminus\{a\}$ and w_2 be the neighbor of w in $V(P_2)\setminus\{c\}$.

If $w_1 \neq b$ and $w_2 \neq d$, there is a $3PC(w,b)$. If $w_1 = b$ and $w_2 \neq d$, there is an odd wheel with center b. So we must have $w_1 = b$ and $w_2 = d$. Hence w is of Type a. □

Definition 2.2 *Let $S_a(\Sigma)$ be the set of nodes adjacent to nodes e and g and a node in $\tilde{P}_1$. Note that, for any node $a' \in S_a(\Sigma)$, there are connected squares Σ' containing a' but not a. When no confusion can occur, we simply write S_a for $S_a(\Sigma)$. The sets $S_c, S_e, S_g, T_b, T_d, T_f, T_h$ are defined analogously. Define S_{ac} to be the set of nodes w such that $N(w)\cap V(\Sigma) = \{e, g\}$. Note that, a node in S_{ac} may replace either a or c in connected squares that contain seven of the nodes a,b,c,d,e,f,g,h. The sets S_{eg}, T_{bd}, T_{fh} are defined analogously. Finally, let $S = S_a \cup S_c \cup S_e \cup S_g \cup S_{ac} \cup S_{eg}$ and $T = T_b \cup T_d \cup T_f \cup T_h \cup T_{bd} \cup T_{fh}$.*

As a consequence of Theorem 2.1, the set S_a is made up by node a and all the Type a[2.1] nodes that are adjacent to e and g and a node in $\tilde{P}_1$. The set S_{ac} is made up by all the Type b[2.1] nodes that are adjacent to e and g. Furthermore the set S is made up by the node set $\{a,c,e,g\}$ and all the nodes that are strongly adjacent to Σ and have two neighbors in $\{a,c,e,g\}$.

Lemma 2.3 *The sets S and T are disjoint and no node of S is adjacent to a node of T.*

Proof: The first property follows immediately from Theorem 2.1. If the second property does not hold, there is a 3-path configuration connecting a node in $\{a,c,e,g\}$ and a node in $\{b,d,f,h\}$. □

Let v be a node in $S_{ac}\cup S_{eg}$ and consider the following classification of the paths in the family $\mathcal{P}_v(\Sigma)$ of direct connections between v and T, avoiding the set $S\setminus\{v\}$. When no confusion arises, we write $\mathcal{P}_v$ instead of $\mathcal{P}_v(\Sigma)$.

Classification 2.4 *Let $P = x_1, x_2, \ldots, x_n$ be a direct connection in $\mathcal{P}_v$ where x_1 is adjacent to v and x_n is adjacent to a node in T.*

- *P is* attached *if x_n is adjacent to a node in $T_b \cup T_d \cup T_f \cup T_h$.*
- *P is* detached *if x_n is not adjacent to any node in $T_b \cup T_d \cup T_f \cup T_h$. Hence x_n is adjacent only to nodes in $T_{bd} \cup T_{fh}$.*

The above classification induces a classification of the strongly adjacent nodes of Type b[2.1]:

Classification 2.5 *Let $v \in S_{ac} \cup S_{eg}$.*

- *Node v is* attached *if v has at least one attached direct connection in $\mathcal{P}_v$.*
- *Node v is* detached *if $\mathcal{P}_v$ is nonempty and all the direct connections in $\mathcal{P}_v$ are detached.*
- *Node v is* separable *if $\mathcal{P}_v$ is empty.*

Similarly, each node $w \in T_{bd} \cup T_{fh}$ is classified as attached, detached, separable, based on the direct connections in $\mathcal{P}_w$ between w and S, avoiding $T \setminus \{w\}$.

In the remainder of this section we study properties of a direct connection $P = x_1, x_2, \ldots, x_n$ in $\mathcal{P}_v$ and we assume that $v \in S_{ac}$ and that x_1 is adjacent to v.

Definition 2.6 *A direct connection $P = x_1, x_2, \ldots, x_n$ in $\mathcal{P}_v$ is* minimal *if, in the subgraph induced by the nodes in $V(P) \cup V(\Sigma)$, no direct connection $P' \in \mathcal{P}_v$ exists, such that*

$$V(P') \setminus V(\Sigma) \subset V(P) \setminus V(\Sigma)$$

Remark 2.7 *The following properties hold for a minimal direct connection in $\mathcal{P}_v$.*

- *If v is detached, then every direct connection in $\mathcal{P}_v$ is minimal.*
- *Let $P = x_1, x_2, \ldots, x_n$ be a minimal direct connection, and let x_j be the node with highest index in $V(P) \setminus V(\Sigma)$. Then no node x_i, $i \leq j-1$ is adjacent to a node in $V(\Sigma) \setminus \{a, c, e, g\}$.*

Lemma 2.8 *Let $v \in S_{ac}$ be an attached node, and let $P = x_1, x_2, \ldots, x_n$ be an attached minimal direct connection in $\mathcal{P}_v$, where x_n is adjacent to a node $t \in T_b \cup T_d \cup T_f \cup T_h$ and x_j is the node of highest index in $V(P) \setminus V(\Sigma)$. Then the following holds:*

(i) *Node t belongs to $T_b \cup T_d$, say $t \in T_b$.*

(ii) *The nodes of $N(x_j) \cap V(\Sigma)$ are contained in P_1.*

(iii) *Node a is adjacent to at most one node x_i, $i \leq j$ and no node x_i, $i \leq j$ is adjacent to a node in the set $\{c, e, g\}$.*

(iv) *Node x_n cannot be adjacent to a node $t \in T_b$ and to a node $t' \in T_d$.*

Proof: Since P is a minimal direct connection in $\mathcal{P}_v$, no node x_l, $1 \leq l \leq i-1$ is adjacent to a node in $V(\Sigma) \setminus \{a, c, e, g\}$. We now divide the proof into the following claims:

Claim 1 *If x_j is strongly adjacent to Σ and is of Type c[2.1], then no node x_i, $i < j$ is adjacent to a node in the set $\{a, c, e, g\}$.*

Proof of Claim 1: Assume that x_j has a neighbor z_1 in $\tilde{P}_1$ and z_2 in $\tilde{P}_2$. Let x_i, $i < j$ be the node of highest index adjacent to a node $x^* \in \{a, c, e, g\}$.

If $x^* = e$, the following three paths induce a $3PC(b, e)$.

$$Q_1 = b, f, P_3, e \quad Q_2 = b, h, P_4, g, c, e \quad ; Q_3 = b, \ldots, z_1, x_j, P_{x_j x_i}, x_i, e$$

Similarly, if $x^* = g$, There is a $3PC(b, g)$.

If $x^* = a$, then z_1 is adjacent to a, else there is a $3PC(z_1, a)$. Now let Q be the shortest path from x_i to e, using intermediate nodes in $V(P_{x_{i-1} x_1}) \cup \{v\}$. Then the hole $H = x_j, P_{x_j x_i}, x_i, Q, e, P_3, f, b, \ldots, z_1, x_j$ induces a wheel with center a.

If $x^* = c$, the proof follows by symmetry and if x_j has a neighbor in $\tilde{P}_3$ and a neighbor in $\tilde{P}_4$, the proof is identical.

Claim 2 *The set $N(x_j) \cap V(\Sigma)$ is contained in one of the sets $V(P_1)$, $V(P_2)$, $V(P_3)$, $V(P_4)$.*

Proof of Claim 2: Assume the contrary holds. Then, by Theorem 2.1 and the fact that $x_j \notin S \cup T$, node x_j is of Type c[2.1]. Assume that node x_j has

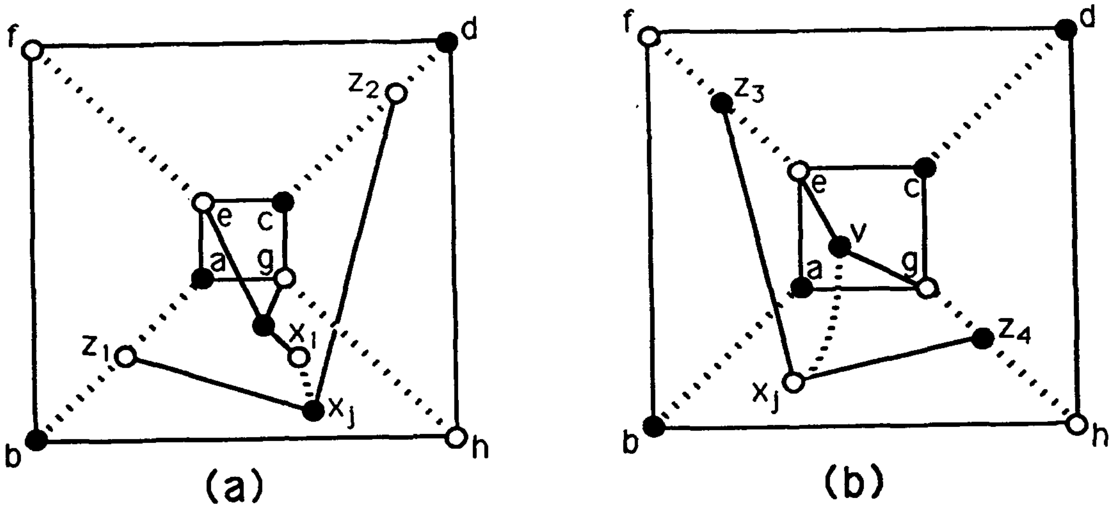

Figure 3:

neighbors $z_1 \in \tilde{P}_1$ and $z_2 \in \tilde{P}_2$. By Claim 1, the following three paths induce a $3PC(e, x_j)$, see Figure 3(a).

$$Q_1 = e, v, x_1, P_{x_1 x_j}, x_j \quad Q_2 = e, a, \ldots, z_1, x_j \quad Q_3 = e, c, \ldots, z_2, x_j$$

Assume now that x_j has neighbors $z_3 \in \tilde{P}_3$ and $z_4 \in \tilde{P}_4$, see Figure 3(b). Then v and x_j are not adjacent, else there is a wheel with center x_j. Now, by Claim 1, there is a $3PC(v, x_j)$.

Claim 3 *Let z be a node in $\{b, d, f, h\}$, $P_i \in \{P_1, P_2, P_3, P_4\}$ be the path whose endnode is z and let $w \in \{a, c, e, g\}$ the other endnode of P_i. If $t \in T_z$, then no node x_l, $l \leq j-1$ can be adjacent to a node of $V(\Sigma) \setminus \{w\}$.*

Proof of Claim 3: Assume that $t \in T_h$ and that P contains a node x_i, $i < j$, adjacent to a node in the set $\{a, c, e\}$. Let x_l be the node of highest index, adjacent to a node in the set $\{a, c, e\}$. As a consequence of Claim 2, the set $N(x_j) \cap V(\Sigma)$ is contained in P_4 (possibly, $N(x_j) \cap V(\Sigma)$ is empty when $j = n$). If x_l is adjacent to a or c there is a $3PC(t, a)$ or a $3PC(t, c)$. If x_l is adjacent to e, there is a $3PC(e, d)$. If $t \in T_b \cup T_d \cup T_f$, the proof is identical.

We now prove Part (i) of Lemma 2.8: Assume $t \in T_f \cup T_h$, say t belongs to T_f, see Figure 4(a). Then by Claim 2, the set $N(x_j) \cap V(\Sigma)$ is contained in P_3 and by Claim 3, no node x_l, $l < j$, is adjacent to a node in the set $\{a, c, g\}$. Then the following three paths induce a $3PC(b, g)$.

$$Q_1 = b, t, P, v, g \quad Q_2 = b, P_1, a, g \quad Q_3 = b, h, P_4, g$$

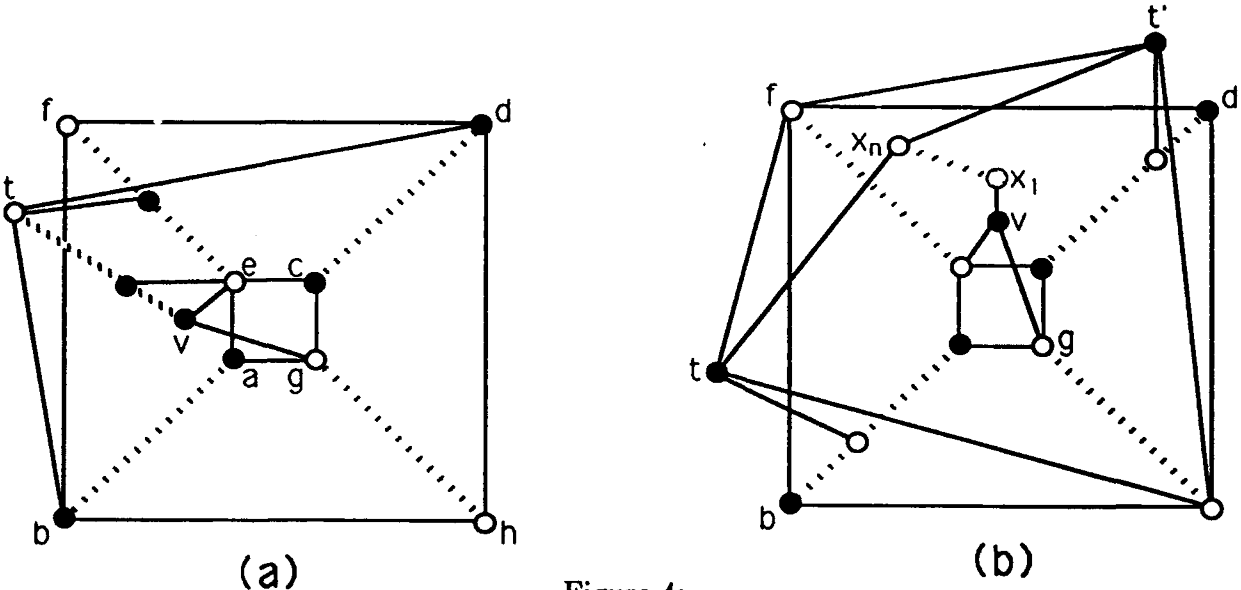

Figure 4:

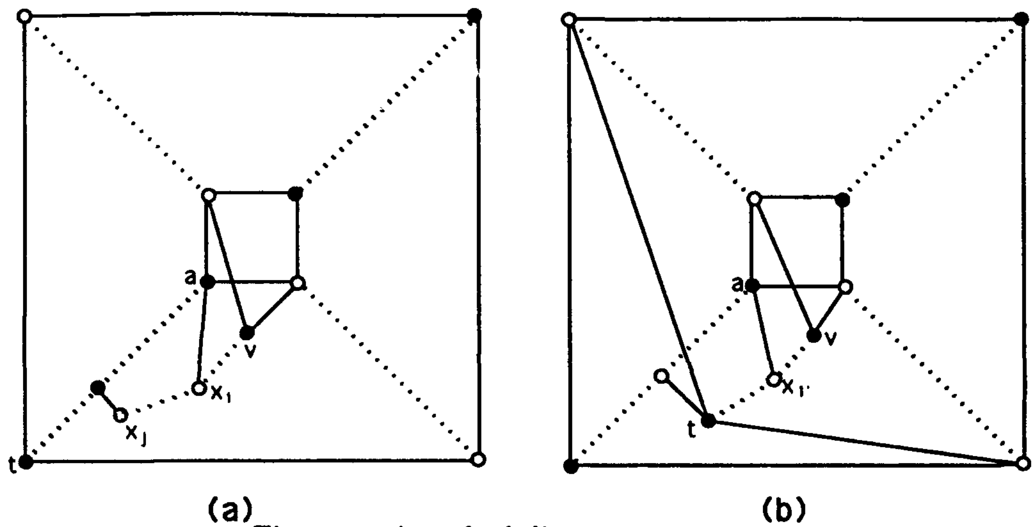

Figure 5: Attached direct connections

Part (ii) now follows from Part (i) and Claim 2. Part (iii) follows from the assumption that the graph contains no wheel and Claim 3.

We finally prove Part (iv). It follows from Part (iii) that no intermediate node of P is adjacent to a node in Σ. This shows the existence of a $3PC(t, g)$, see Figure 4(b). □

Remark 2.9 *Lemma 2.8 shows that, up to symmetry, Figure 5 depicts all the possible attached direct connections in $\mathcal{P}_v$, where, in Figure 5(a), node a may not be adjacent to a node x_i of P and node x_j may have two neighbors in P_1.*

We now characterize detached direct connections in $\mathcal{P}_v$, where v is a detached node.

Lemma 2.10 *Let $P = x_1, x_2, \ldots, x_n$ be a direct connection in $\mathcal{P}_v$, where x_1 is adjacent to a detached node $v \in S_{ac}$ and x_n is adjacent to a node $t \in T_{bd} \cup T_{fh}$. Then P satisfies the following properties;*

- *No node x_i, $1 \leq i \leq n$ is adjacent to a node in Σ.*
- *Node t belongs to T_{bd}.*

Proof: Since v is a detached node, then no node x_i, $1 \leq i \leq n$ is adjacent to a node in $V(\Sigma) \setminus \{a, c, e, g\}$. Let x_l be the node with highest index adjacent to a (unique) node in the set $\{a, c, e, g\}$. Assume $t \in T_{bd}$. If x_l is adjacent to a or c, there is a $3PC(a, f)$ or a $3PC(c, f)$. If x_l is adjacent to e or g, there is a $3PC(e, t)$ or a $3PC(g, t)$. If $t \in T_{fh}$, the proof is identical. Hence the first part of the lemma follows. The second part now follows immediately, for, if $t \in T_{fh}$, there is a $3PC(b, g)$. $\square$

3 Bicliques in Connected Squares

Definition 3.1 *Consider the following node sets, associated to connected squares Σ:*

- $S'(\Sigma) = S_a \cup S_c \cup S_e \cup S_g \cup \{x \in S_{ac} \cup S_{eg} : x$ *is attached* $\} \cup \{x \in S_{ac} : x$ *is detached* $\}$. *When no confusion arises, we write S' instead of $S'(\Sigma)$.*
- $S'' = S_a \cup S_c \cup S_e \cup S_g \cup \{x \in S_{ac} \cup S_{eg} : x$ *is attached* $\} \cup \{x \in S_{eg} : x$ *is detached*$\}$.
- $T' = T_b \cup T_d \cup T_f \cup T_h \cup \{x \in T_{bd} \cup T_{fh} : x$ *is attached* $\} \cup \{x \in T_{bd} : x$ *is detached*$\}$.
- $T'' = T_b \cup T_d \cup T_f \cup T_h \cup \{x \in T_{bd} \cup T_{fh} : x$ *is attached* $\} \cup \{x \in T_{fh} : x$ *is detached*$\}$.
- $S^* = S' \cup S'' = S \setminus \{x \in S_{ac} \cup S_{eg} : x$ *is separable*$\}$.
- $T^* = T' \cup T'' = T \setminus \{x \in T_{bd} \cup T_{fh} : x$ *is separable*$\}$.

We denote by $K_{S'}(\Sigma)$, $K_{S''}(\Sigma)$, $K_{T'}(\Sigma)$, $K_{T''}(\Sigma)$, $K_{S^*}(\Sigma)$, $K_{T^*}(\Sigma)$ the subgraphs of G induced by the above node sets. Again, when no confusion is possible, we write $K_{S'}$ instead of $K_{S'}(\Sigma)$.

The goal of this section is to prove the following theorem:

Theorem 3.2 *Each of the subgraphs $K_{S'}$, $K_{S''}$, $K_{T'}$, $K_{T''}$ is a biclique in connected squares Σ.*

Furthermore, K_{S^*} is a biclique if and only if K_{T^*} is a biclique.

The following lemmas show that new connected squares can be obtained from Σ by replacing two paths from $\{P_1, P_2, P_3, P_4\}$ by (attached or detached) minimal direct connections. All the combinations of pairs of paths needed for the proof of Theorem 3.2 will be considered in the lemmas.

Definition 3.3 *Let x be a node in $S_a \setminus \{a\}$. Then x belongs to a unique connected squares Σ^*, such that $V(\Sigma^*) \setminus V(\Sigma) = \{x\}$. Connected squares Σ^* is said to be obtained from Σ by* substitution of node *a with node x. When $x = a$, it will be convenient to write, by extension, that $\Sigma^* = \Sigma$ is obtained by substitution of node a with node x.*

Let $v \in S_{ac}(\Sigma)$ be an attached node, having minimal attached direct connection $P = x_1, x_2, \ldots, x_n$ in $\mathcal{P}_v$, where x_n is adjacent to a node $t \in T_b(\Sigma)$. Then Lemma 2.8 shows the existence of connected squares $\Sigma^* = CS(v, P, t,\ P_2,\ P_3,\ P_4)$. Then Σ^* is said to be obtained from Σ by substitution of path P_1 *with v, P, t.*

Remark 3.4 *Let $v \in S_{ac}(\Sigma)$ and $t \in T_{bd}(\Sigma)$ be two detached nodes linked by a detached direct connection P in $\mathcal{P}_v$. Then Lemma 2.10 shows the existence of connected squares $\Sigma^* = CS(v, P, t,\ P_2,\ P_3,\ P_4)$ and $\Sigma^{**} = CS(P_1, v, P, t, P_3, P_4)$ obtained from Σ by substituting respectively P_1 with v, P, t and P_2 with v, P, t.*

Furthermore P is a direct connection in both $\mathcal{P}_v$ and $\mathcal{P}_t$.

Lemma 3.5 *Let $u \in S_a(\Sigma) \cup S_c(\Sigma) \cup T_b(\Sigma) \cup T_d(\Sigma)$ and $v \in S_e(\Sigma) \cup S_g(\Sigma) \cup T_f(\Sigma) \cup T_h(\Sigma)$. W.l.o.g. assume $u \in S_a(\Sigma)$ and $v \in S_e(\Sigma) \cup T_f(\Sigma)$.*

Let Σ_u be the connected squares obtained from Σ by substituting node a with u. Then connected squares Σ_{uv} can be obtained from Σ_u by substituting a node of Σ_u with v. Furthermore, if Σ_{vu} is defined by substituting first node v and then node u, then Σ_{uv} and Σ_{vu} coincide.

Proof: We show that u and v are adjacent if and only if both u and v belong to either $S(\Sigma)$ or $T(\Sigma)$. If $v \in S_e(\Sigma)$, then u and v are adjacent, else there is a $3PC(c, f)$. If $v \in T_f(\Sigma)$, then u and v are adjacent, by Lemma 2.3. Now the proof follows. □

Lemma 3.6 *Let u be a node in $S_a(\Sigma)$ and P be a minimal direct connection in $\mathcal{P}_v$ between an attached node $v \in S_{eg}(\Sigma)$ and a Type a[2.1] node $w \in T_f(\Sigma)$. Let Σ_u be obtained from Σ by substituting a with u and Σ_v be obtained from Σ by substituting P_3 with v, P, w. Then u can be substituted for a in Σ_v and v, P, w can be substituted for P_3 in Σ_u and the two connected squares thus obtained coincide.*

Proof: We show that u is adjacent to v and u is not adjacent to a node in $\{w\} \cup V(P)$.

By Lemma 2.3, nodes u and w are nonadjacent. Node u is adjacent to v and is not adjacent to a node of P, otherwise u is a strongly adjacent node in Σ_v, not satisfiying Theorem 2.1. □

Lemma 3.7 *Let P be a minimal direct connection in $\mathcal{P}_u$ between an attached node $u \in S_{ac}(\Sigma)$ and $t \in T_b(\Sigma)$. Let Q be a minimal direct connection in $\mathcal{P}_v$ between an attached node $v \in S_{eg}(\Sigma)$ and $w \in T_f(\Sigma)$. Let Σ_u be obtained from Σ by substituting P_1 with u, P, t. Let Σ_v be obtained from Σ by substituting P_3 with v, Q, w.*

Then v, Q, w can be substituted for P_3 in Σ_u and u, P, t can be substituted for P_1 in Σ_v and the two connected squares thus obtained coincide.

Proof: We show that u and v are adjacent, t and w are adjacent, and no other adjacency exists, between nodes in $\{u, t\} \cup V(P)$ and $\{v, w\} \cup V(Q)$. We first prove the following claim:

Claim *Nodes t and w are adjacent. Node t is not adjacent to a node in $V(Q) \cup \{v\}$. Node w is not adjacent to a node in $V(P) \cup \{u\}$.*

Proof of Claim: The first part of the claim follows from Lemma 3.5.

Assume that t is adjacent to a node in $V(Q) \cup \{v\}$. Then t is a strongly adjacent node in Σ_v, violating Theorem 2.1. The proof of the claim is now complete by symmetry.

Now node u cannot have a neighbor in $V(Q) \cup \{w\}$, else u is a strongly adjacent node in Σ_v, violating Theorem 2.1. Similarly, v cannot have a neighbor in $V(P) \cup \{t\}$.

Let Σ^* be the connected squares obtained by substituting b with t and f with w.

No node of P is adjacent to a node of Q, else u or v is attached in Σ^*, having a minimal direct connection in $\mathcal{P}_u$ or $\mathcal{P}_v$ not satisfying Lemma 2.8. Finally u and v are adjacent, else there is a $3PC(e, t)$. □

Lemma 3.8 *Let u be an attached node in $S_{ac}(\Sigma)$, having minimal direct connection P in $\mathcal{P}_u$ to a node $t \in T_b(\Sigma)$. Let $v \in S_{eg}(\Sigma)$ be a detached node, having direct connection Q in $\mathcal{P}_v$ to a node $w \in T_{fh}(\Sigma)$. Let Σ_u be obtained from Σ by substituting P_1 with u, P, t. Let Σ_v be obtained from Σ by substituting P_3 with v, Q, w. Then v, Q, w can be substituted for P_3 in Σ_u and u, P, t can be substituted for P_1 in Σ_v and the two connected squares thus obtained coincide.*

Proof: Again, we show that u and v are adjacent, t and w are adjacent also, and no other adjacency exists between nodes in $\{u, t\} \cup V(P)$ and $\{v, w\} \cup V(Q)$.

Let Σ_t be the connected squares obtained fron Σ by substituting b with t.

Then nodes t and v are not adjacent, else v is a strongly adjacent node in Σ_t, violating Theorem 2.1. If t is not adjacent to w or if t is adjacent to a node of Q, then node v is an attached node in Σ_t, having minimal direct connection in $\mathcal{P}_v$ violating Lemma 2.8. Hence t is adjacent to w and t is not adjacent to any node in $\{v\} \cup V(Q)$.

Now u is adjacent to v, and no other adjacency exists, between the nodes in $V(P) \cup \{u\}$ and $V(Q) \cup \{v, w\}$, else w is an attached node in Σ_u, having an attached minimal direct connection in $\mathcal{P}_w$ not satisfying Lemma 2.8. □

Lemma 3.9 *Let u be a node in $S_a(\Sigma)$. Let $v \in S_{eg}(\Sigma)$ be a detached node, having direct connection P in $\mathcal{P}_v$ to a node $w \in T_{fh}(\Sigma)$. Let Σ_u be obtained from Σ by substituting a with u. Let Σ_v be obtained from Σ by substituting P_3 with v, P, w. Then v, P, w can be substituted for P_3 in Σ_u and u can be substituted for a in Σ_v and the two connected squares thus obtained coincide.*

Proof: If u is adjacent to w, node w violates Theorem 2.1 in Σ_u. If u is adjacent to P, the node w has an attached direct connection, violating Lemma 2.8. If u is not adjacent to v, there is a $3PC(b, e)$. □

Lemma 3.10 *Let u be a detached node in $S_{ac}(\Sigma)$, having direct connection P in $\mathcal{P}_u$ to a node $t \in T_{bd}(\Sigma)$. Let $v \in S_{eg}(\Sigma)$ be a detached node, having direct connection Q in $\mathcal{P}_v$ to a node $w \in T_{fh}(\Sigma)$. Let Σ_u be obtained from Σ by substituting P_1 with u, P, t. Let Σ_v be obtained from Σ by substituting P_3 with v, Q, w.*

(i) *If u and v are adjacent, v, Q, w can be substituted for P_3 in Σ_u and u, P, t can be substituted for P_1 in Σ_v and the two connected squares thus obtained coincide.*

(ii) *If u and v are nonadjacent, no adjacency exists between the nodes in $\{u, t\} \cup V(P)$ and $\{v, w\} \cup V(Q)$.*

Proof: By Lemma 2.3, nodes u and w are nonadjacent. If w is adjacent to P, there is a detached direct connection in $\mathcal{P}_w$ which violates Lemma 2.10. By symmetry, t is not adjacent to $\{v\} \cup V(Q)$.

If u and v are adjacent, then t is adjacent to w, else there is a $3PC(c, u)$. This proves Part (i).

If u and v are nonadjacent, then t and v are nonadjacent, else there is a $3PC(b, t)$. This proves Part (ii). □

Proof of Theorem 3.2: First we show that $K_{S'}$ is a biclique.

Let $u \in S_e \cup S_g$. Then u is adjacent to every node in $S_a \cup S_c$ by Lemma 3.5, to every attached node in S_{ac} by Lemma 3.6 and to every detached node in S_{ac} by Lemma 3.9.

Let $u \in S_{eg}$. Then u is adjacent to every attached node in S_{ac} by Lemma 3.7 and to every detached node in S_{ac} by Lemma 3.8.

This shows that $K_{S'}$ is a biclique. By symmetry, $K_{S''}$, $K_{T'}$ and $K_{T''}$ are bicliques. The last statement of the theorem follows from Lemma 3.10. □

4 A Property of Bicliques

Theorem 4.1 *There exist connected squares Σ^* whose induced subgraphs K_{S^*} and K_{T^*} are both bicliques.*

Proof: In this proof, when we say that nodes x and y are linked by a direct connection $\tilde{P}$, we define $P = x, \tilde{P}, y$. Let $\Sigma^0 = CS(P_1^0, P_2^0, P_3^0, P_4^0)$ be connected squares with $P_1^0 = a^0, \ldots, b^0$, $P_2^0 = c^0, \ldots, d^0$, $P_3^0 = e^0, \ldots, f^0$, $P_4^0 = g^0, \ldots, h^0$. If $K_{S^*}(\Sigma^0)$ is not a biclique then, by Theorem 3.2 and Lemma 3.10, there exist one pair of detached nodes $a^1 \in S_{a^0c^0}(\Sigma^0)$, and $b^1 \in T_{b^0d^0}(\Sigma^0)$ linked by a direct connection $\tilde{P}_1^1$ in $\mathcal{P}_{a^1}(\Sigma^0)$and another pair of detached nodes $g^1 \in S_{e^0g^0}(\Sigma^0)$ and $h^1 \in T_{g^0h^0}(\Sigma^0)$ linked by a direct connection $\tilde{P}_4^1$ in $\mathcal{P}_{g^1}(\Sigma^0)$ such that no adjacency exists between the nodes of P_1^1 and P_4^1, see Figure 6.

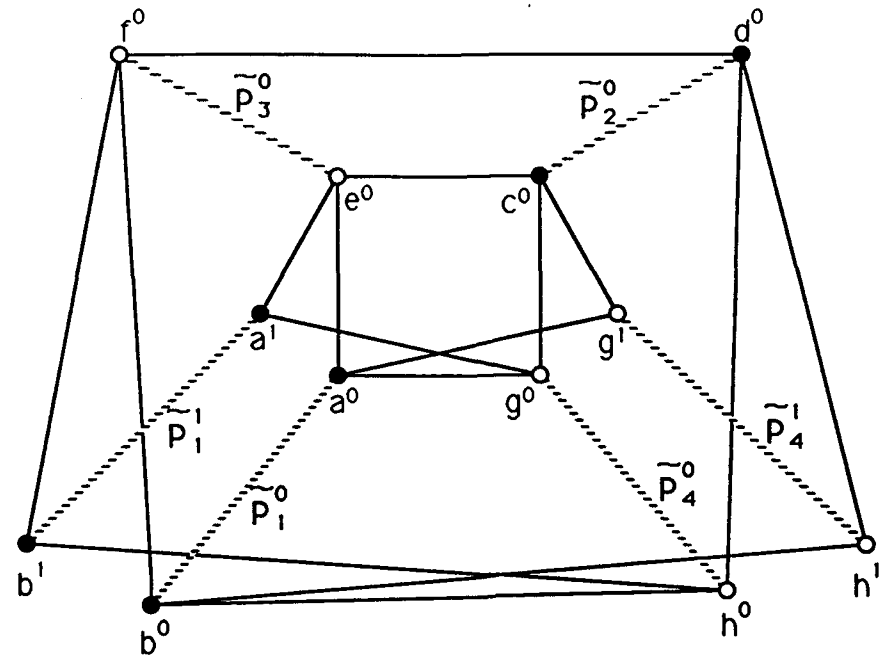

Figure 6:

Let Σ^1 be the connected squares obtained by substituting in Σ^0 the path P_1^0 with the path P_1^1. If $K_{S^*}(\Sigma^1)$ is not a biclique, then in Σ^1 there exist a detached node $a^2 \in S_{a^1c^0}(\Sigma^1)$, having direct connection $\tilde{P}_1^2$ in $P_{a^2}(\Sigma^I)$ to a detached node $b^2 \in T_{b^1d^0}(\Sigma^1)$ and a detached node $g^2 \in S_{e^0g^0}(\Sigma^1)$, having direct connection $\tilde{P}_4^2$ in $P_{g^2}(\Sigma^I)$ to a detached node $h^2 \in T_{f^0h^0}(\Sigma^1)$ such that no adjacency exists between the nodes of P_1^2 and P_4^2. Note that, at this stage, we are not ruling out $a^2 = a^0$.

By Lemma 3.10, the subgraph induced by $V(\Sigma^1) \cup V(P_1^2) \cup V(P_4^2)$ has no other adjacencies except the ones shown in Figure 7.

We now show that the configuration of Figure 8 is induced, that is, the only adjacencies of P_1^0 and P_4^1 with the subgraph of Figure 7 are depicted in Figure 8. In other words, we need to establish the adjacencies between $V(P_4^2)$ and $V(P_4^1)$, between $V(P_4^2)$ and $V(P_1^0)$, between $V(P_1^2)$ and $V(P_4^1)$ and between $V(P_1^2)$ and $V(P_1^0)$.

Note that $h^2 \neq h^1$, since h_2 is adjacent to b_1 but h_1 is not. Furthermore $h^2 \neq h^0$, since h^0 is adjacent to b^2 but h^2 is not. The same argument shows that $b^2 \neq b^1$, $g^2 \neq g^1$, $g^2 \neq g^0$ and $a^2 \neq a^1$.

Claim 1 *Node g^2 is not adjacent to any node in $V(P_1^0) \cup V(P_4^1) \setminus \{a^0\}$.*

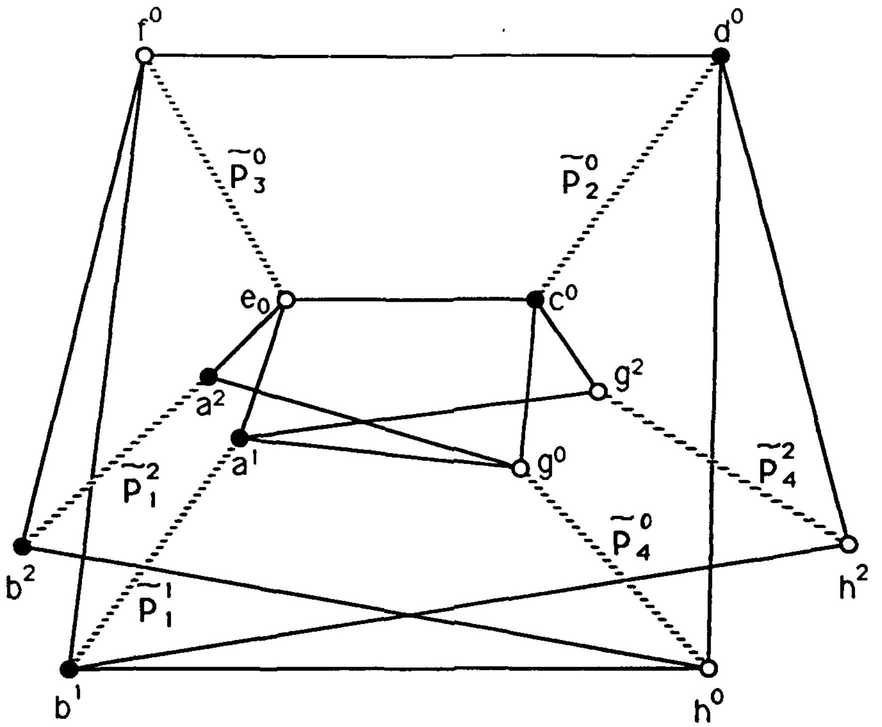

Figure 7:

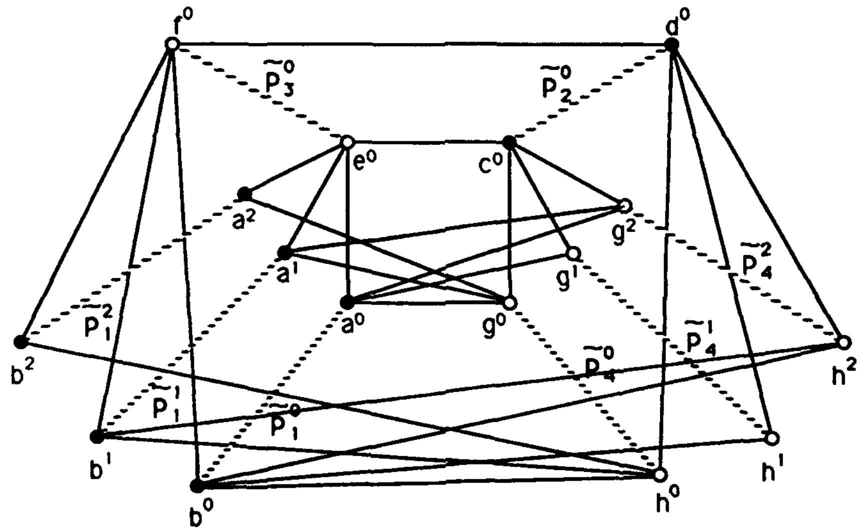

Figure 8:

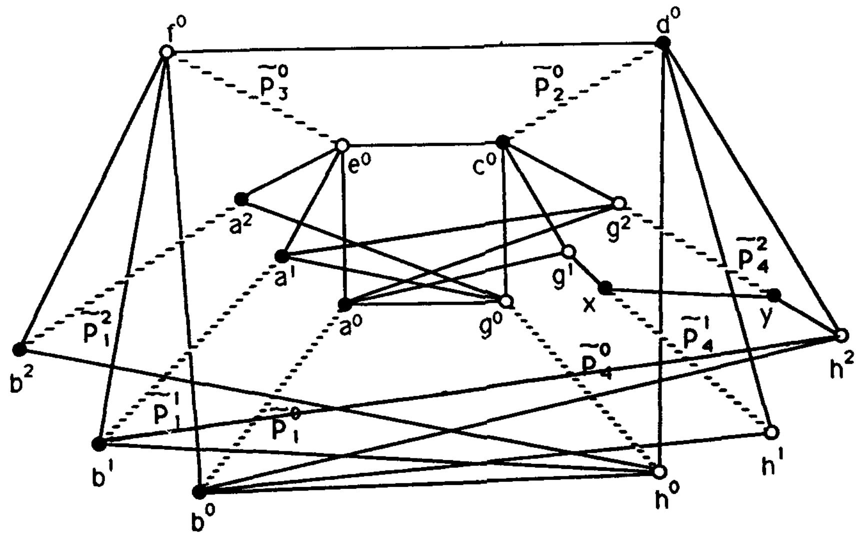

Figure 9:

Proof of Claim 1: Let Σ_1 be the connected squares obtained by substituting in Σ^0 the path P_4^0 with P_4^1. Assume that node g^2 has a neighbor in P_4^1. Then g^2 is a strongly adjacent node in Σ_1 and by Theorem 2.1 node $g^2 \in S_{g^1}(\Sigma_1)$. Hence g^2 is adjacent to a^0. Let Σ_1^* be the connected squares obtained by substituting g^1 with g^2 in Σ_1. Theorem 2.1 applied to Σ_1^* shows that g^1 either belongs to $S_{g^2}(\Sigma_1^*)$ or is an attached node in $S_{e^0g^2}(\Sigma_1^*)$. Since a^1 is a detached or an attached node in Σ_1^*, Lemmas 3.6-3.9 applied to Σ_1^* show that a^1 and g^1 are adjacent, a contradiction.

Finally, node g^2 cannot have a neighbor in $V(P_1^0) \setminus \{a^0\}$, otherwise g^2 is a strongly adjacent node in Σ^0, violating Theorem 2.1.

By symmetry, the above proof shows the following:

Claim 2 *Node h^2 is not adjacent to any node in $V(P_1^0) \cup V(P_4^1) \setminus \{b^0\}$.*

Claim 3 *No node of $\tilde{P}_4^2$ is adjacent to or coincident with a node of P_4^1.*

Proof of Claim 3: Claims 1 and 2 show that no node of $\tilde{P}_4^1$ is adjacent to g^2 or h^2. Let $x \in V(P_4^1)$ and $y \in V(\tilde{P}_4^2)$ be two adjacent or coincident nodes such that the length of the g^1x-subpath of P_4^1 is minimized and the length of the h^2y-subpath of P_4^2 is minimized. Then since g^1 and a^1 are nonadjacent, the following three paths induce a $3PC(e^0, b^1)$, see Figure 9.

$$Q_1 = e^0, c^0, g^1, \ldots, x, y, \ldots, h^2, b^1 \quad Q_2 = e^0, a^1, P_1^1, b^1 \quad Q_3 = e^0, P_0^3, f^0, b^1$$

Claim 4 *Nodes g^2 and a^0 are adjacent and nodes h^2 and b^0 are adjacent. Nodes g^2 and b^0 are nonadjacent and nodes h^2 and a^0 are nonadjacent.*

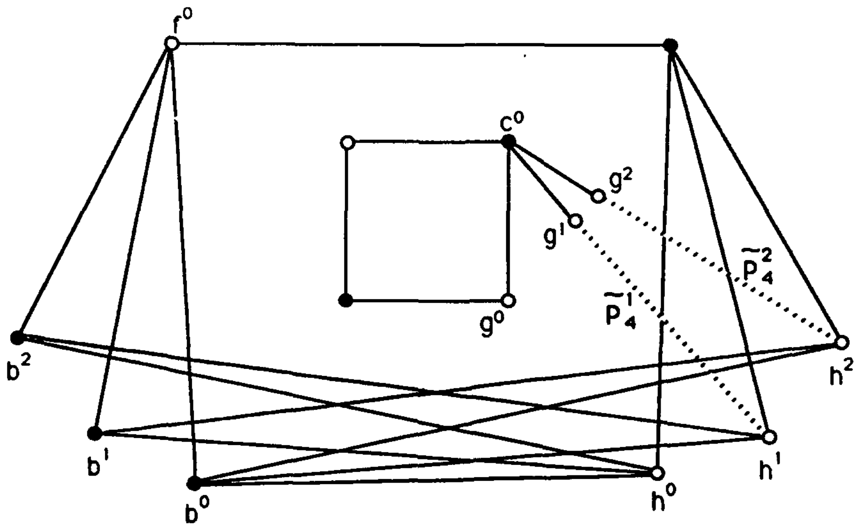

Figure 10:

Proof of Claim 4: Since nodes g^2, h^2, a^0 and b^0 are attached or detached in connected squares Σ^1, Theorem 3.2 shows that nodes g^2 and a^0 are adjacent if and only if nodes h^2 and b^0 are adjacent. Assume that g^2 and a^0 are nonadjacent. Then the following three paths induce a $3PC(e^0, d^0)$:

$$Q_1 = e^0, P_3^0, f^0, d^0, \quad Q_2 = e^0, a^0, g^1, P_4^1, h^1, d^0, \quad Q_3 = e^0, a^1, g^2, P_4^2, h^2, d^0$$

If g^2 and b^0 are adjacent, then g^2 is a strongly adjacent node in Σ^0, violating Theorem 2.1. By symmetry, the proof is now complete.

Note that Claim 4 implies that $a^2 \neq a^0$ and $b^2 \neq b^0$.

Claim 5 *No node of $\tilde{P}_4^2$ is adjacent to or coincident with a node of P_1^0.*

Proof of Claim 5: Assume not. Then in Σ^0, node g^2 or h^2 is an attached node having a minimal direct connection in $\mathcal{P}_{g^2}(\Sigma^0)$ or in $\mathcal{P}_{h^2}(\Sigma^0)$ violating Lemma 2.8.

Claim 6 *Nodes a^2 and b^2 are not adjacent to any node in P_4^1.*

Proof of Claim 6: Let Σ_1 be the connected squares defined in the proof of Claim 1. Node b^2 can only be adjacent to h^1 in P_4^1, otherwise b^2 is a strongly adjacent node in Σ_1, not satisfying Theorem 2.1. If b^2 is adjacent to h^1, consider the chordless cycle $H = b^2, h^1, P_4^1, g^1, c^0, g^2, P_4^2, h^2, b^1, f^0, b^2$, see Figure 10. Then (H, b^0) is an odd wheel. The proof for a^2 follows by symmetry.

Claim 7 *Nodes a^2 and b^2 are not adjacent to any node in P_1^0.*

Proof of Claim 7: Assume by contradiction that node a^2 has a neighbor in P_1^0. Then node a^2 belongs to $S_a(\Sigma^0)$. Now Lemma 3.9 applied to Σ_0 shows that a^2 and g^2 are adjacent, a contradiction.

Claim 8 *No node of $\tilde{P}_1^2$ is adjacent to a node in P_1^0.*
Proof of Claim 8: Assume not. Then nodes a^2 and b^2 are attached nodes in Σ^0. Again, Lemma 3.8 applied to Σ^0 shows that a^2 and g^2 or b^2 and h^2 are adjacent, a contradiction.

Claim 9 *No node of $\tilde{P}_1^2$ is adjacent to a node in P_4^1.*
Proof of Claim 9: Assume not. Then nodes a^2 or b^2 are detached nodes in Σ^0 , having minimal direct connections in $\mathcal{P}_{a^2}(\Sigma^0)$ or in $P_{b^2}(\Sigma^0)$ violating Lemma 2.10.

Claims 1-9 show that the graph of Figure 8 is induced.

Starting from Σ^0, we construct a sequence of connected squares $\Sigma^1, \Sigma^2, \ldots, \Sigma^{n-1}, \Sigma^n$ as follows:

If $K_{S^*}(\Sigma^{i-1})$ and $K_{T^*}(\Sigma^{i-1})$ are not bicliques, there exist two pairs of non-adjacent nodes a^i, b^i and g^i, h^i that are detached in Σ^{i-1} and have detached direct connections $\tilde{P}_1^i$ and $\tilde{P}_4^i$ in $\mathcal{P}_{a^i}(\Sigma^{i-1})$ and in $\mathcal{P}_{g^i}(\Sigma^{i-1})$ respectively.

Connected squares Σ^i are obtained by substituting in Σ^{i-1} the path P_1^{i-1} with P_1^i. Consider now the following property:

Property 10 *Every Σ^i, $0 \leq i \leq n$, satisfies the following:*

10.1 *Node h^i is adjacent to d^0 and to the nodes b^j, $0 \leq j \leq i-1$*

10.2 *Node g^i is adjacent to c^0 and to the nodes a^j, $0 \leq j \leq i-1$*

10.3 *Node a^i is adjacent to e^0, g^0 and to no node g^j, $1 \leq j \leq i$*

10.4 *Node b^i is adjacent to h^0, f^0 and to no node h^j, $1 \leq j \leq i$*

10.5 *No node of $V(\tilde{P}_1^i) \cup V(\tilde{P}_4^i)$ is adjacent to a node in the set*

$$\bigcup_{1 \leq j \leq 4} V(P_j^0) \bigcup_{1 \leq k \leq i-1} V(P_1^k) \cup V(P_4^k)$$

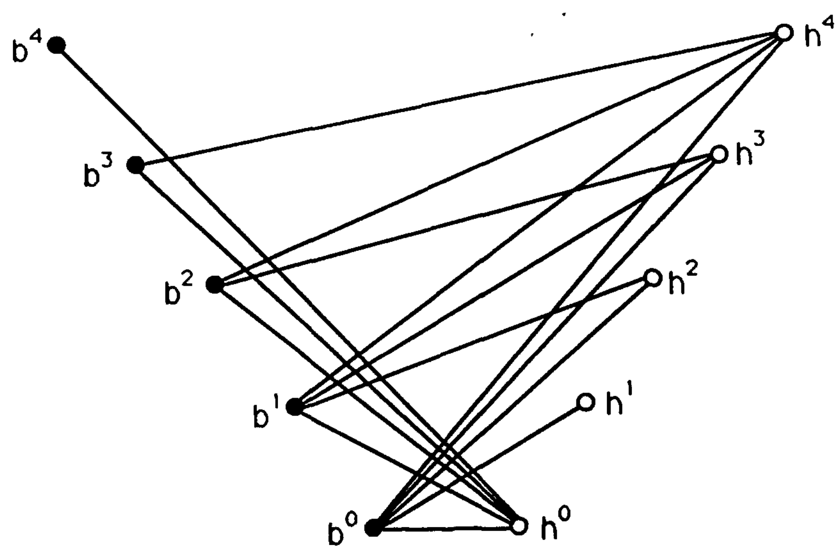

Figure 11:

Claims 1-9 show that Property 10 holds for Σ^0, Σ^1 and Σ^2, and Figure 11 shows the adjacencies, according to Property 10, between nodes b^l and h^l, $0 \leq l \leq 4$.

Assume that Σ^{i-1} does not satisfy the theorem. Hence with respect to Σ^{i-1}, there exist two pairs of nonadjacent detached nodes a^i, b^i and g^i, h^i having direct connections $\tilde{P}_1^i$ and $\tilde{P}_4^i$ in $\mathcal{P}_{a^i}(\Sigma^{i-1})$ and in $P_{g^i}(\Sigma^{i-1})$ respectively. We inductively assume that Property 10 holds for $n = i-1$ and we show the following:

Claim 11 *The nodes in the set* $V(P_1^i) \cup V(P_4^i)$ *satisfy Property 10.*

Proof of Claim 11: The above inductive hypothesis shows that for all indices l, m such that $0 \leq l < m \leq i-1$, the pairs of nodes a^m, b^m and g^m, h^m constitute two pairs of nonadjacent detached nodes in Σ^l, having detached direct connections $\tilde{P}_1^m$ and $\tilde{P}_4^m$ in $\mathcal{P}_{a^m}(\Sigma^l)$ and in $\mathcal{P}_{g^m}(\Sigma^l)$ respectively. Hence Σ^m is also obtained from Σ^l by substituting P_1^l with P_1^m.

This implies that nodes a^{i-1}, b^{i-1} and g^{i-1}, h^{i-1} constitute two pairs of nonadjacent detached nodes with respect to Σ^l, for all $0 \leq l < i-1$. Hence the graph G^l, induced by the node set

$$V(\Sigma^l) \cup V(P_1^{i-1}) \cup V(P_4^{i-1}) \cup V(P_1^i) \cup V(P_4^i)$$

is isomorphic to the graph of Figure 8 induced by the node set $V(\Sigma^0) \cup V(P_1^1) \cup V(P_4^1) \cup V(P_1^2) \cup V(P_4^2)$. Hence by applying Claims 1-9 to G^l, we have that Properties 10.1, 10.2 hold. Furthermore Properties 10.3 and 10.4

hold for $j = i-1$, and Property 10.5 holds except for the adjacencies between nodes of $V(\tilde{P}_1^i) \cup V(\tilde{P}_4^i)$ and the nodes in the set $\bigcup_{1\leq j\leq i-2} V(P_4^j)$.

This implies that nodes a^i, b^i and g^i, h^i constitute a pair of nonadjacent detached nodes with respect to Σ^j, for all $0 \leq j \leq i-1$. Hence by applying *Claims 1-9 to the graph induced by the nodes* $V(\Sigma^0) \cup V(P_1^j) \cup V(P_4^j) \cup V(P_1^i) \cup V(P_4^i)$ we have that no node of $V(\tilde{P}_1^i) \cup V(\tilde{P}_4^i)$ is adjacent to a node in the set $\bigcup_{1\leq j\leq i-2} V(P_4^j)$. This completes the proof of Property 10.5. Furthermore Properties 10.3 and 10.4 hold for all $1 \leq j \leq i-2$. This proves Claim 11.

The proof of the theorem is now complete by finiteness of the graph, since an unlimited sequence of connected squares $\Sigma^0, \ldots, \Sigma^i, \ldots, \Sigma^n$ implies an unlimited growth in the size of the node set of the graph. □

5 Biclique Cutsets and 2-Joins

Throughout this section we assume that connected squares $\Sigma = CS(P_1, P_2, P_3, P_4)$ satisfy Theorem 4.1. That is, both subgraphs K_{S^*} and K_{T^*} are bicliques.

Theorem 5.1 *If connected squares Σ contain a separable node v, then K_{S^*} or K_{T^*} is a biclique cutset, separating v from Σ.*

Proof: By definition, no direct connection between v and T avoids $S \setminus \{v\}$. Let x_k be the node of $S \setminus \{v\}$ with highest index in a direct connection $P = x_1, x_2, \ldots, x_n$ between v and T. Then x_k either belongs to $S_a \cup S_c \cup S_e \cup S_g$ or is an attached or detached node in $S_{ac} \cup S_{eg}$. Hence x_k belongs to S^*. □

We now further assume that connected squares Σ contain no separable node. Hence $S = S^*$ and $T = T^*$. We define $G^*(V, E^*)$ to be the partial subgraph obtained from $G(V, E)$ by removing the edge set $E(K_S) \cup E(K_T)$.

Definition 5.2 *Let $S^c = S \cap V^c$, $S^r = S \cap V^r$ and $T^c = T \cap V^c$, $T^r = T \cap V^r$. Let W^c be the set of nodes which belong to at least one minimal direct connection in $\mathcal{P}_v$ from a node v in $S^c \cup T^c$ and let $Z^c = W^c \cup S^c \cup T^c$. Similarly, let W^r be the subset of nodes in at least one minimal direct connection from a node in $S^r \cup T^r$ and let $Z^r = W^r \cup S^r \cup T^r$.*

Lemmas 3.5-3.10 show the following:

Remark 5.3 *For every pair of nodes $u \in Z^c$ and $v \in Z^r$, there exist the following connected squares:*

Σ^u *with the following properties:*

- *Connected squares Σ^u are obtained from $\Sigma = CS(P_1, P_2, P_3, P_4)$ by substituting at most one path P_1 or P_2 with a path P^u.*

- *Node u belongs to P^u.*

Σ^v *with the following properties:*

- *Connected squares Σ^v are obtained from $\Sigma = CS(P_1, P_2, P_3, P_4)$ by substituting at most one path P_3 or P_4 with a path P^v.*

- *Node v belongs to P^v.*

Σ^{uv} *with the following properties:*

- *Connected squares Σ^{uv} are obtained from $\Sigma = CS(P_1, P_2, P_3, P_4)$ by substituting at most one path P_1 or P_2 with a path P^u and at most one path P_3 or P_4 with a path P^v.*

- *Node u belongs to P^u and node v belongs to P^v.*

Lemma 5.4 *The node sets Z^c and Z^r satisfy the following properties:*

- *In $G^*(V, E^*)$, no node of Z^c is adjacent to or coincident with a node of Z^r.*

- *In $G^*(V, E^*)$, no node $w \notin Z^c \cup Z^r$ is adjacent to a node in Z^c and a node in Z^r.*

Proof: The first part of the lemma follows directly from Remark 5.3.

Assume that a node $w \notin Z^c \cup Z^r$ is adjacent to a node u in Z^c and a node v in Z^r. Node w is not strongly adjacent to Σ, else $w \in S(\Sigma) \cup T(\Sigma)$, contradicting the assumption $w \notin Z^c \cup Z^r$. Let $\Sigma^u = CS(P^u, P_2, P_3, P_4)$, $\Sigma^v = CS(P_1, P_2, P^v, P_4)$ and $\Sigma^{uv} = CS(P^u, P_2, P^v, P_4)$ be connected squares defined in Remark 5.3 where w.l.o.g. we assume that P_1 is substituted with P^u and P_3 is substituted with P^v. Theorem 2.1 shows that node w is a Type a[2.1] node in Σ^{uv}. This shows that w is a strongly adjacent node in Σ^u or in Σ^v, violating Theorem 2.1. $\square$

Theorem 5.5 *Let Σ be a connected squares. If $E(K_S) \cup E(K_T)$ is not a 2-join of G and neither K_S nor K_T is a biclique cutset of G, then there exists a path $P = x_1, x_2, \ldots, x_n$, $n > 1$ with at least one of the following properties:*

- *The path P is a direct connection between $Z^c \setminus S^c$ and $Z^r \setminus T^r$, avoiding $S^c \cup T^r$ such that no node x_i, $1 < i < n$ is adjacent to a node in T^r.*
- *The path P is a direct connection between $Z^c \setminus S^c$ and $Z^r \setminus T^r$, avoiding $S^c \cup T^r$ such that no node x_i, $1 < i < n$ is adjacent to a node in S^c.*
- *The path P is a direct connection between $Z^c \setminus T^c$ and $Z^r \setminus S^r$, avoiding $T^c \cup S^r$ such that no node x_i, $1 < i < n$ is adjacent to a node in T^c.*
- *The path P is a direct connection between $Z^c \setminus T^c$ and $Z^r \setminus S^r$, avoiding $T^c \cup S^r$ such that no node x_i, $1 < i < n$ is adjacent to a node in S^r.*

Proof: By Lemma 5.4 no node of Z^c is adjacent to or coincident with a node of Z^r. Hence since $E(K_S) \cup E(K_T)$ is not a 2-join, there exists in $G^*(V, E^*)$ a direct connection $P = x_1, x_2, \ldots, x_n$ between Z^c and Z^r, where x_1 is adjacent to a node in Z^c and x_n is adjacent to a node in Z^r. Furthermore Lemma 5.4 shows that $n > 1$.

If $(N(x_1) \cup N(x_n)) \cap (Z^c \cup Z^r) \not\subseteq S$ and $(N(x_1) \cup N(x_n)) \cap (Z^c \cup Z^r) \not\subseteq T$, then P belongs to at least one of the above four families of direct connections and we are done. So assume w.l.o.g. that $(N(x_1) \cup N(x_n)) \cap (Z^c \cup Z^r) \subseteq S$, that is, the set $N(x_1) \cap (Z^c \cup Z^r)$ is contained in S^c and the set $N(x_n) \cap (Z^c \cup Z^r)$ is contained in S^r.

Since K_S is not a biclique cutset, separating P from $Z^c \cup Z^r$, there exists a direct connection $Q = y_1, y_2, \ldots, y_m$ between $V(P)$ and $Z^c \cup Z^r$ and avoiding S, where y_1 is adjacent to a node in $V(P)$ and y_m is adjacent to a node in $Z^c \cup Z^r$. Note that for all $1 < i < m$, we have that $N(y_i) \cap (Z^c \cup Z^r) \subset S$. Since Lemma 5.4 shows that y_m cannot be adjacent to a node in Z^c and a node in Z^r, we assume w.l.o.g. that $N(y_m) \cap (Z^c \cup Z^r) \subseteq Z^r$ and $N(y_m) \cap (Z^c \cup Z^r) \setminus S \neq \emptyset$.

If some node of Q is adjacent to a node in S^c, let $y_i \neq y_m$ be such a node with highest index. Then the $y_i y_m$-subpath of Q is a direct connection between $Z^c \setminus T^c$ and $Z^r \setminus S^r$, avoiding $T^c \cup S^r$. Note that by construction, an intermediate node of such subpath can not be adjacent to a node in T^c. Hence the theorem follows.

If no node y_i, $1 < i < m$ in Q is adjacent to a node in S^c, let x_j be the node of P, adjacent to $y_1 \in V(Q)$ such that the length of the $x_1 x_j$-subpath P_{1j} of P is the shortest. Then the path $R = x_1, P_{1j}, x_j, y_1, Q, y_m$ is a direct connection between $Z^c \setminus T^c$ and $Z^r \setminus S^r$, avoiding $T^c \cup S^r$ such that no intermediate node in R is adjacent to a node in T^c. □

We now assume w.l.o.g. that $P = x_1, x_2, \ldots, x_n$, $n > 1$ is a direct connection between $Z^c \setminus S^c$ and $Z^r \setminus T^r$, avoiding $S^c \cup T^r$ such that no node x_i, $1 < i < n$ is adjacent to a node in T^r.

Lemma 5.6 *If x_1 is adjacent to a node of $V(\Sigma) \setminus \{a, c\}$, let u be such a neighbor of x_1. Otherwise let u be a neighbor of x_1 in $Z^c \setminus S^c(\Sigma)$.*

If x_n is adjacent to a node of $V(\Sigma) \setminus \{f, h\}$ let v be such a neighbor of x_n. Otherwise let v be a neighbor of x_n in $Z^r \setminus T^r(\Sigma)$.

Let $\Sigma^u = CS(P^u, P_2, P_3, P_4)$, $\Sigma^v = CS(P_1, P_2, P^v, P_4)$, $\Sigma^{uv} = CS(P^u, P_2, P^v, P_4)$ be connected squares obtained from $\Sigma = CS(P_1, P_2, P_3, P_4)$ as in Remark 5.3, where we assume w.l.o.g. that P_1 is substituted with P^u and P_3 is substituted with P^v. Then the following holds:

(i) *Either x_1 is a Type c[2.1] node in Σ^{uv} and $\Sigma^u = \Sigma$ (i.e. the path P^u coincides with P_1), or the set $N(x_1) \cap V(\Sigma^{uv})$ is contained in P^u.*

(ii) *Either x_n is a Type c[2.1] node in Σ^{uv} and $\Sigma^v = \Sigma$ (i.e. the path P^v coincides with P_3), or the set $N(x_n) \cap V(\Sigma^{uv})$ is contained in P^v.*

(iii) *The set $N(x_i) \cap V(\Sigma^{uv}) \subseteq \{a^u, c\}$ for every node x_i, $1 < i < n$ of P, where $a^u = V(P^u) \cap S$.*

Proof: We prove Part (i). Assume $N(x_1) \cap (V(\Sigma) \setminus \{a, c\}) \neq \emptyset$. Then x_1 cannot be a Type a[2.1] or a Type b[2.1] strongly adjacent node to Σ, else $x_1 \in S(\Sigma) \cup T(\Sigma)$. Hence Theorem 2.1 shows that either the set $N(x_1) \cap V(\Sigma)$ is contained in $P_1 = P^u$ or x_1 is a Type c[2.1] strongly adjacent node to Σ and, by construction, $\Sigma^u = \Sigma$.

Assume $N(x_1) \cap (V(\Sigma) \setminus \{a, c\}) = \emptyset$ and $N(x_1) \cap \{a, c\} = \{a\}$ or $\{c\}$, say $N(x_1) \cap \{a, c\} = \{a\}$. Let $u \in N(x_1) \cap (Z^c \setminus S^c(\Sigma))$ be a node belonging to a direct connection R in $\mathcal{P}_w$ between a node $w \in S(\Sigma)$ and $t \in T(\Sigma)$. If P_1 can be substituted with w, R, t, then (i) follows. If P_1 cannot be substituted with w, R, t, then P_2 can be substituted with w, R, t.

Let Σ^u be the connected squares obtained with the above substitution. Then x_1 is a strongly adjacent node in Σ^u, violating Theorem 2.1.

Finally, if $N(x_1) \cap (V(\Sigma) \setminus \{a, c\}) = \emptyset$ and $N(x_1) \cap \{a, c\} = \emptyset$, by construction, node x_1 has no neighbors in P_2^u. This completes the proof of Part (i).

Part (ii) follows by symmetry. Now, by assumption, for every node x_i, $1 < i < n$, the set $N(x_i) \cap (Z^c \cup Z^r)$ is contained in $S^c(\Sigma)$ and $S^c(\Sigma) \cap V(\Sigma^{uv}) = \{a^u, c\}$. This proves Part (iii). □

Theorem 5.7 *The graph induced by $V(\Sigma^{uv}) \cup V(P)$ is not signable to be balanced.*

Proof: We consider the following cases:

Case 1 *The path P contains a node x_i, $1 < i < n$ adjacent to a^u and c^u.*

Proof of Case 1: Let x_i be such a node with lowest index. Then x_i is an attached node in Σ^{uv}, having the $x_{i-1}x_1$-subpath of P as attached minimal direct connection in $\mathcal{P}_{x_i}$. However this minimal direct connection violates Lemma 2.8.

Case 2 *The path P contains no node x_i adjacent to both a^u and c and node x_1 or node x_n is of Type c[2.1] in Σ^{uv}.*

Proof of Case 2: Assume that x_1 is of Type c[2.1] and assume w.l.o.g. that x_1 has a neighbor z_1 in P^u and a neighbor z_2 in P_2. The same argument used in the proof of Claim 1 of Lemma 2.8 shows that if a node x_i, $1 < i < n$ is adjacent to a^u, then z_1 and a^u are adjacent. If x_i and c are adjacent, then z_2 and c are adjacent. Finally, z_1 is not adjacent to a^u or z_2 is not adjacent to c.

If x_n is a Type c[2.1] node in Σ^{uv}, having neighbors z_3 in $\tilde{P}^v$ and z_4 in $\tilde{P}_4$, then no node x_i, $1 < i < n$ is adjacent to a^u or c, else there is a $3PC(x_n, a^u)$ or a $3PC(x_n, c)$. Hence there is a $3PC(z_1, z_3)$.

If x_n is not a Type c[2.1] in Σ^{uv}, assume w.l.o.g. that z_2 and c are not adjacent. Then there is a $3PC(z_2, c)$. Hence x_1 cannot be a Type c[2.1] node.

The same argument shows that x_n cannot be a Type c[2.1] node.

Case 3 *The path P contains no node x_i adjacent to both a^u and c, and neither node x_1 nor node x_n is of Type c[2.1].*

Proof of Case 3: By Lemma 5.6, we can assume w.l.o.g. that $N(x_1) \subset V(P^u)$ and that $N(x_n) \subset V(P^v)$. Let a^u and b^u denote the endnodes of P^u and e^v, f^v the endnodes of P^v.

If there exists a node x_i, $1 < i < n$ adjacent to c, let x_j be such a node with lowest index and let u' be the node of P^u adjacent to x_1, such that the length of the $u'b^u$-subpath P^* of P^u is shortest. Then the following three paths induce a $3PC(c,h)$:

$$Q_1 = c, x_j, \ldots, x_1, u', P^*, b^u, h; \;\; Q_2 = c, g, P_4, h; \;\; Q_3 = c, P_2, d, h$$

If no node x_i, $1 < i < n$ is adjacent to c, there is again a $3PC(c,h)$. □

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In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G:

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1,K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1,K_2 disconnects G.
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G, where S can be partitioned into three disjoint sets T,A,N such that ~~$T\neq\emptyset$~~, some node ~~$a\in$~~ T is adjacent to every node in $A\cup N$ and, if $|T|\geq 2$, then $|A|\geq$ ~~2 and every~~ node of T is adjacent to every node of A.

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r,V^c;E)$ as follows: The node nets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij}=1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.